# Singular Integrals on $(-\infty, \infty)$
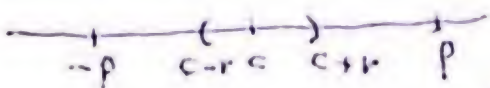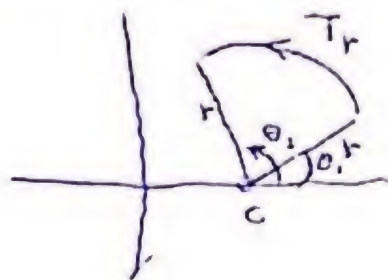
Let $f : (-\infty, \infty) \to \mathbb{C}$ be a continuous function except at $c_1, c_2, \ldots, c_N$. For simplicity, assume that $f$ has a singularity at $c$ only



Then
$$pv \int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{\rho \to \infty \\ r \to 0+}} \left\{ \int_{-\rho}^{c-r} f(x)dx + \int_{c+r}^{\rho} f(x)dx \right\}.$$

Lemma Let $w = f(z)$ be a complex-valued function with a simple pole at $c$.



$T_r : z = c + re^{i\theta}, \quad \theta_1 \leq \theta \leq \theta_2$.

Then
$$\lim_{r \to 0} \int_{T_r} f(z)dz = i(\theta_2 - \theta_1) \, Res(f, c).$$

Proof  The Laurent series is
$$f(z) = \frac{a_{-1}}{z - c} + \underbrace{\sum_{n=0}^{\infty} a_n(z - c)^n}_{g(z)}, \quad 0 < |z - c| < R.$$

So, for $r < R$,
$$\int_{T_r} f(z)dz = a_{-1} \int_{T_r} \frac{1}{z - c}dz + \int_{T_r} g(z)dz.$$

Now,
$$a_{-1} \int_{T_r} \frac{1}{z - c}dz = a_{-1} \int_{\theta_1}^{\theta_2} \frac{1}{re^{i\theta}} ire^{i\theta}d\theta$$
$$= i(\theta_2 - \theta_1) \, Res(f, c).$$

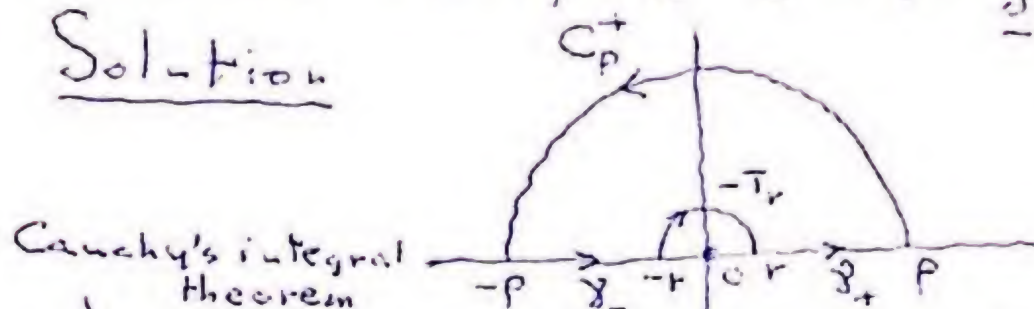Next, suppose $|g(z)| \leq M$ for all $z$ with $|z-c| < R_1$, $R_1 < R$. Then

$$\int_{T_r} |g(z)| \, dz \leq M(\theta_2 - \theta_1) r \to 0$$

as $r \to 0$. $\therefore$

$$\lim_{r \to 0} \int_{T_r} g(z) \, dz = i(\theta_2 - \theta_1) \operatorname{Res}(g, c).$$

<u>Example</u> Compute $I = pv \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx$.

<u>Solution</u>



Cauchy's integral theorem

$\Gamma_{\rho,r} = C_\rho^+ - \gamma_- - T_r + \gamma_+$

$$0 = \int_{\Gamma_{\rho,r}} \frac{e^{iz}}{z} \, dz = \int_{C_\rho^+} \frac{e^{iz}}{z} \, dz + \int_{-\rho}^{-r} \frac{e^{iz}}{z} \, dz - \int_{T_r} \frac{e^{iz}}{z} \, dz$$
$$+ \int_{r}^{\rho} \frac{e^{iz}}{z} \, dz$$

Let $\rho \to \infty$, $r \to 0+$. Then

$$0 = \lim_{\substack{\rho \to \infty \\ r \to 0+}} \left\{ \int_{-\rho}^{-r} \frac{e^{ix}}{x} \, dx + \int_{r}^{\rho} \frac{e^{ix}}{x} \, dx \right\}$$
$$- \lim_{r \to 0+} \int_{T_r} \frac{e^{iz}}{z} \, dz$$

$\therefore pv \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dz = i\pi \lim_{z \to 0} e^{iz} = i\pi$

$\therefore pv \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi.$